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ON THE DYNAMIC STABILITY OF THE COURNOT DUOPOLY SOLUTION UNDER BOUNDED RATIONALITY

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RESUMEN: La mayoría de los modelos de oligopolio descritos en la literatura analizan los procesos dinámicos y la estabilidad del equilibrio de Nash mediante la introducción de especificaciones concretas para las funciones de demanda y de costes. Este trabajo analiza la estabilidad dinámica del equilibrio de Cournot-Nash en el contexto de un duopolio utilizando funciones generales para describir tanto la demanda como los costes. Se concluye que la condición que garantiza la estabilidad del equilibrio de Nash bajo el proceso de ajuste implícito en el modelo original de Cournot es un requisito clave en la estabilidad dinámica del equilibrio de Cournot-Nash independientemente del esquema de expectativas de las empresas. Además, esta condición es más decisiva cuanto mayor sea el grado de racionalidad de las empresas.

Palabras claves: Duopolio no lineal, expectativas, equilibrio de Cournot-Nash, estabilidad dinámica.

ABSTRACT: The most of the oligopolistic models described in the existing literature analyze dynamic processes and the stability of the Nash equilibrium by introducing concrete specifications for the demand and the cost functions. This paper analyzes the dynamic stability of the Cournot-Nash equilibrium in the context of a duopoly using general functions to establish both demand and costs. The condition that guarantees the stability of the Nash equilibrium under the adjustment process implicit in the Cournot's original model is found to be a key requirement underpinning the dynamic stability of the Cournot-Nash equilibrium regardless of the firms' expectations scheme. Moreover, this condition is more decisive the higher the degree of rationality of firms.

Keywords Nonlinear duopoly, expectations, Cournot-Nash equilibrium, dynamic stability.

1. Introduction

The model proposed by Cournot (1838) is the cornerstone of modern oligopoly theory. Its analysis has given rise to an extensive and long-standing discussion in the literature, mainly concerning the uniqueness and stability of the equilibrium (see the relevant contributions made by Hahn 1962, Okuguchi 1964, 1976, Friedman 1977 and Dixit 1986, among others).

However, the Cournot model has also been criticized. Thus, one unsatisfactory aspect is the way in which the market reaches equilibrium. As Martin (1993) observes, an ad hoc dynamic process is grafted onto a static model, and it is assumed that quantities will converge at the Nash equilibrium point through a process of continuous adjustment based on the reaction functions of the agents. Similarly, as Corchón and Mas-Colell (1996) point out, Cournot (1838) defines the term of best reply dynamics and conjectures that, in the case of oligopoly with homogeneous product, an equilibrium would be reached asymptotically.

The subsequent analysis of best reply dynamics has run through two alternative approaches: one based on the assumption of continuous time and the other based on discrete time.

From the perspective of continuous time it has been shown that under the assumptions of homogeneous product, strategic substitution and additional conditions on the slopes of demand and cost functions, the dynamics converge asymptotically to equilibrium (see Hahn 1962, Okuguchi 1964).

Contributions based on discrete time have shown that under certain strong hypotheses, the best response dynamics converge asymptotically to equilibrium (see Friedman 1977, Vives 1990, Milgrom and Roberts 1990, among others). Different expectations to those based on best reply dynamics have been used in the case of duopoly, concluding that, in general, the possibility of chaotic behavior cannot be ruled out (see Dana and Montrucchio 1986, 1987).

In recent years, there has been a growing interest in the dynamic analysis of competition in oligopoly markets. In this line, in a context of discrete time, several schemes of expectations have been defined according to the degree of rationality of the firms. In this regard, it has been shown that these sophisticated expectation rules produce complex dynamic behavior, demonstrating that Cournot oligopoly dynamics may never converge to equilibrium and can result in chaotic behavior in the long term (Puu 1991; Kopel 1996; Agiza 1998, 1999; Andaluz et al., 2020).

However, the conclusions obtained in most of the existing contributions depend fundamentally on the precise specifications defining the market structure, such as the demand and the cost functions. Therefore, the objective of this paper is to answer the following question: what can be said about dynamic stability of the Cournot-Nash equilibrium in a discrete-time duopoly model as general as possible, without the need to introduce specific functional forms of demand and costs?

We show that the sufficient condition that guarantees the stability of the Nash equilibrium under the adjustment process implicit in Cournot's original model is revealed as a fundamental condition in the dynamic stability of the Nash equilibrium in the dynamics process resulting from expectations associated with different degrees of rationality.

The remainder of this paper is organized as follows. Section 2 presents the static Cournot duopoly. Section 3 then goes on to develop a dynamic duopoly under different expectations rules, and Section 4 closes the paper with our main conclusions.

2. Static duopoly

We consider a quantity-setting duopoly where firms produce a homogeneous product. Denoting the quantity produced by firm i (i = 1, 2) as q_i, we assume a decreasing inverse demand function $P = P(q) = P(q_1 + q_2)$, while the firm's cost functions $C_i(q_i)$ are increasing and convex. These functions are at least of class $C^{(2}$. Each firm will produce the quantity that maximizes its profit function, which is given by:

$$\left.\begin{array}{l}\Pi_1(q_1,q_2) = P(q_1+q_2)\,q_1 - C_1(q_1)\\ \Pi_2(q_1,q_2) = P(q_1+q_2)\,q_2 - C_2(q_2)\end{array}\right\}. \tag{1}$$

From the first order maximization condition (marginal profits equal to zero), we obtain the best response functions or reaction curves for each firm:

$$\left.\begin{aligned} \frac{\partial \Pi_1(q_1,q_2)}{\partial q_1} = 0 \Rightarrow P(q_1+q_2)+P'(q_1+q_2)q_1 - C_1'(q_1) = 0 \Rightarrow q_1 = R_1(q_2) \\ \frac{\partial \Pi_2(q_1,q_2)}{\partial q_2} = 0 \Rightarrow P(q_1+q_2)+P'(q_1+q_2)q_2 - C_2'(q_2) = 0 \Rightarrow q_2 = R_2(q_1) \end{aligned}\right\} \quad (2)$$

In line with Hahn (1962), we assume that the marginal profit for each firm is a diminishing function of its rival's output. Formally:

$$\frac{\partial^2 \Pi_i(q_1,q_2)}{\partial q_i \partial q_j} = P'(q_1+q_2)+P''(q_1+q_2)q_i < 0, \ \ i \neq j, \ i, j = 1,2 . \quad (3)$$

The fulfilment of (3) and the convexity of cost functions guarantee the fulfilment of the second order condition of maximum:

$$\frac{\partial^2 \Pi_i(q_1,q_2)}{\partial q_i^{\,2}} = 2P'(q_1+q_2)+P''(q_1+q_2)q_i - C_i''(q_i) < 0, \qquad i = 1,2 . \quad (4)$$

Therefore, the intersection of the reaction curves given in (2) defines the Cournot-Nash equilibrium, $E^* = \left(q_1^*, q_2^*\right)$.

Moreover, (3) implies that the reaction curves defined in (2) exhibit negative slope, given that $R_i'\left(q_j\right) = -\dfrac{\frac{\partial^2 \Pi_i(q_1,q_2)}{\partial q_i \partial q_j}}{\frac{\partial^2 \Pi_i(q_1,q_2)}{\partial q_i^{\,2}}} < 0, \text{ for } i \neq j.$ Thus, quantities are strategic substitute variables (see Bulow et al. 1985).

Based on the sequential reactions of each firm to the observed output of the other, an implicit adjustment process is defined whereby the quantities of the two firms may converge to the Cournot-Nash equilibrium. In this context (see Martin 1993), the Cournot-Nash equilibrium $E^* = \left(q_1^*,\ q_2^*\right)$ will be stable if[1]:

$$\left|R_1'\left(q_2^*\right)R_2'\left(q_1^*\right)\right| < 1 \Leftrightarrow \left|\frac{\partial^2 \Pi_1^*}{\partial q_1^{\,2}}\ \frac{\partial^2 \Pi_2^*}{\partial q_2^{\,2}}\right| > \left|\frac{\partial^2 \Pi_1^*}{\partial q_1 \partial q_2}\ \frac{\partial^2 \Pi_2^*}{\partial q_2 \partial q_1}\right|$$

Under the assumptions established for the demand and cost functions, and condition (3) the above condition is equivalent to the following:

$$\frac{\partial^2 \Pi_1^*}{\partial q_1^{\,2}}\ \frac{\partial^2 \Pi_2^*}{\partial q_2^{\,2}} - \frac{\partial^2 \Pi_1^*}{\partial q_1 \partial q_2}\ \frac{\partial^2 \Pi_2^*}{\partial q_2 \partial q_1} > 0 . \quad (5)$$

3. Dynamic duopoly

Under the assumption of discrete time scale, different decision rules have been introduced. Namely, naïve expectations and adaptive expectations are found as representations of dynamic adjustment procedure based on the best response functions. The relaxation of the assumption that firms have full knowledge of

[1] Π_i^* denotes that the profit is evaluated at the equilibrium point $E^* = \left(q_1^*,\ q_2^*\right)$.

demand leads to the definition of other adjustment mechanisms that involve a lesser degree of rationality, such as the Local Monopolistic Approximation (LMA) (see Bischi et al 2007), and the gradient rule based on marginal profit (see Bischi et al 2010).

In this section we will show that condition (5) also plays a decisive role in the dynamic stability of the Cournot-Nash equilibrium under these expectations rules.

3.1. *Naïve expectations*

According to this decision rule firms expect that the rival's output in each period will be equal to the last observable quantity. The dynamic system is thus[2]:

$$T_N : \begin{cases} q_{1,t+1} = R_1(q_{2,t}) \\ q_{2,t+1} = R_2(q_{1,t}) \end{cases} \qquad (6)$$

The stationary point of (6) is obtained when $q_{i,t+1} = q_{i,t} = q_i,\ i = 1,2,$ leading to the Cournot-Nash equilibrium, $E^* = \left(q_1^*, q_2^*\right)$.

In the two-dimensional case of discrete time systems, the condition for local stability of the equilibrium is that the eigenvalues of the corresponding Jacobian matrix evaluated in this point should be less than one in absolute terms (see Gandolfo 2010). We can formulate the following proposition regarding the local stability of the Nash equilibrium:

Proposition 1 *In a dynamic duopoly where firms adopt naïve expectations, the condition (5) is a necessary and sufficient condition for the local stability of the Cournot-Nash equilibrium*.

Proof

The Jacobian matrix of (6) evaluated in E^* is $JT_N\left(q_1^*,\ q_2^*\right) = \begin{pmatrix} 0 & R_1'\left(q_2^*\right) \\ R_2'\left(q_1^*\right) & 0 \end{pmatrix}$ and the eigenvalues are $\lambda_1 = -\sqrt{R_1'\left(q_2^*\right) R_2'\left(q_1^*\right)}$ and $\lambda_2 = \sqrt{R_1'\left(q_2^*\right) R_2'\left(q_1^*\right)}$, which have modulus less than one if, and only if, $R_1'\left(q_2^*\right) R_2'\left(q_1^*\right) < 1.$ □

3.2. *Adaptive expectations*

Let us now turn to firms adopting the best response dynamic with inertia. Each firm *i* changes its output quantity proportionally to the difference between its naïve expectations value, given by the reaction function $R_i(q_j)$, and the quantity for the last period. Formally:

$$q_{i,t+1} - q_{i,t} = \beta_i\left(R_i(q_{j,t}) - q_{i,t}\right), i \neq j, i, j = 1,2, \ \text{ with } 0 < \beta_i \leq 1$$

From the previous expression we obtain the dynamic system[3]:

$$T_A : \begin{cases} q_{1,t+1} = (1-\beta_1)q_{1,t} + \beta_1 R_1(q_{2,t}) \\ q_{2,t+1} = (1-\beta_2)q_{2,t} + \beta_2 R_2(q_{1,t}) \end{cases} \qquad (7)$$

[2] This model will be linear if the best response functions are linear (which depends on the demand and cost structure).
[3] The model will also be linear in this case if the reaction functions are linear.

The parameter β_i measures firms' reluctance to change the production level chosen previous period in view of the signal given by the reaction function. Note that the naïve expectations case is obtained where $\beta_i = 1$.

We can find the steady states of (7) by solving the following system:

$$\left. \begin{array}{l} q_1 = (1-\beta_1)q_1 + \beta_1 R_1(q_2) \Rightarrow q_1 = R_1(q_2) \\ q_2 = (1-\beta_2)q_2 + \beta_2 R_2(q_1) \Rightarrow q_2 = R_2(q_1) \end{array} \right\}$$

Therefore, the Nash equilibrium point $E^* = \left(q_1^*, q_2^*\right)$ will be the same as in (6).

In the two-dimensional case of discrete time systems, the condition for the local stability of the equilibrium can be expressed in terms of the trace (*Tr*) and determinant (*Det*) of the Jacobian matrix of dynamic system evaluated at the equilibrium point, giving the following inequalities (*Schur's conditions*; see Gandolfo 2010):

$$\left. \begin{array}{l} (i)\ 1 - Tr + Det > 0 \\ (ii)\ 1 + Tr + Det > 0 \\ (iii)\ 1 - Det > 0 \end{array} \right\} \qquad (8)$$

If any single inequality in (8) becomes an equality while the other two are simultaneously fulfilled, the equilibrium loses stability through a transcritical bifurcation when $1 - Tr + Det = 0$, a flip bifurcation when $1 + Tr + Det = 0$, or a Neimark-Sacker bifurcation when $1 - Det = 0$.

We may formulate the following proposition with regard to the local stability of the Nash equilibrium point, $E^* = \left(q_1^*, q_2^*\right)$:

Proposition 2 *In a dynamic duopoly where firms adopt adaptive expectations, the condition (5) is a necessary and sufficient condition for the local stability of the Cournot-Nash equilibrium for* $0 < \beta_i \le 1,\ i = 1, 2$.

Proof The Jacobian matrix of T_A evaluated at $E^* = \left(q_1^*, q_2^*\right)$ is:

$$JT_A\left(E^*\right) = \begin{pmatrix} 1-\beta_1 & \beta_1 R_1'\left(q_2^*\right) \\ \beta_2 R_2'\left(q_1^*\right) & 1-\beta_2 \end{pmatrix}$$

and its trace and determinant are:

$$Tr = Tr\left(JT_A\left(E^*\right)\right) = 2 - \left(\beta_1 + \beta_2\right)$$

$$Det = Det\left(JT_A\left(E^*\right)\right) = 1 - \left(\beta_1 + \beta_2\right) + \beta_1\beta_2\left[1 - R_1'\left(q_2^*\right)R_2'\left(q_1^*\right)\right] =$$

$$= Tr - 1 + Z_A,\ \text{being } Z_A = \beta_1\beta_2\left[1 - R_1'\left(q_2^*\right)R_2'\left(q_1^*\right)\right] > 0 \ \text{ if, and only if, (5) is true.}$$

Substituting these expressions into (8), we can deduce that Schur's conditions are verified if we assume condition (5):

$$\left. \begin{array}{l} (i)\ 1 - Tr + Det = Z_A > 0 \\ (ii)\ 1 + Tr + Det = 2Tr + Z_A = 4 - 2\left(\beta_1 + \beta_2\right) + \beta_1\beta_2\left[1 - R_1'\left(q_2^*\right)R_2'\left(q_1^*\right)\right] > 0 \\ (iii)\ 1 - Det = 2 - \left(Tr + Z_A\right) = \beta_1 + \beta_2 + \beta_1\beta_2\left[1 - R_1'\left(q_2^*\right)R_2'\left(q_1^*\right)\right] > 0 \end{array} \right\}$$

Therefore, the condition (5) is a necessary and sufficient condition for the local stability of E^*. □

We may note here that the local stability of the Nash equilibrium does not depend on the values of parameters β_i.

3.3. *Local Monopolistic Approximation*

A premise of conducting adaptive expectations is that all firms possess a full knowledge concerning with the inverse demand function. This assumption can be partially relaxed under the Local Monopolistic Approximation (LMA) expectation (see Tuinstra 2004, Bischi et al 2007).

Under this expectation rule, given the inverse demand function $P = P(q) = P(q_1 + q_2)$, each firm is able to get a correct estimate of the partial derivative $\frac{\partial P(q)}{\partial q_i} = \frac{\partial P(q_1+q_2)}{\partial q_i} = P'(q_1+q_2) = P'(q)$ in any time period.

This estimate is then used to obtain a computation of the expected price in period *t*+1:

$$P^e_{t+1} = P(q_{1,t} + q_{2,t}) + P'\left(q_{1,t} + q_{2,t}\right)\left(q_{i,t+1} - q_{i,t}\right)$$

Therefore, the expected profit in period *t*+1 is given as:

$$\Pi^e_{i,t+1} = \left[P(q_{1,t} + q_{2,t}) + P'(q_{1,t} + q_{2,t})\left(q_{i,t+1} - q_{i,t}\right)\right] q_{i,t+1} - C_i(q_{i,t+1}),\ i = 1,2$$

From the first order condition for the maximum with respect to $q_{i,t+1}$, we obtain[4]:

$$\frac{\partial \Pi^e_{i,t+1}}{\partial q_{i,t+1}} = 0 \Leftrightarrow P(q_t) + 2P'(q_t)\ q_{i,t+1} - P'(q_t)\ q_{i,t} - C_i'(q_{i,t+1}) = 0,\ i = 1,2 \qquad (9)$$

where $q_t = q_{1t} + q_{2t}$.

Equations in (9) define implicitly the dynamic system:

$$T_{LMA} : \begin{cases} q_{1,t+1} = \phi_1(q_{1,t}, q_{2,t}) \\ q_{2,t+1} = \phi_2(q_{1,t}, q_{2,t}) \end{cases} \qquad (10)$$

It may be deduced that the unique interior fixed point of this dynamic system is the Nash equilibrium $E^* = \left(q_1^*, q_2^*\right)$. To investigate the local stability of the Nash equilibrium, we consider the Jacobian matrix of system (10) evaluated in this point. By implicitly deriving the equations given in (9) it is obtained:

$$JT_{LMA}\left(E^*\right) = \begin{pmatrix} \frac{\partial \phi_1\left(q_1^*,\ q_2^*\right)}{\partial q_1} & \frac{\partial \phi_1\left(q_1^*,\ q_2^*\right)}{\partial q_2} \\ \frac{\partial \phi_2\left(q_1^*,\ q_2^*\right)}{\partial q_1} & \frac{\partial \phi_2\left(q_1^*,\ q_2^*\right)}{\partial q_2} \end{pmatrix} = \begin{pmatrix} -\frac{P''(q^*)\, q_1^*}{2P'(q^*) - C_1''\left(q_1^*\right)} & -\frac{P'(q^*) + P''(q^*)\, q_1^*}{2P'(q^*) - C_1''\left(q_1^*\right)} \\ -\frac{P'(q^*) + P''(q^*)\, q_2^*}{2P'(q^*) - C_2''\left(q_2^*\right)} & -\frac{P''(q^*)\, q_2^*}{2P'(q^*) - C_2''\left(q_2^*\right)} \end{pmatrix}$$

Taking into account (3) and (4), we obtain:

[4] The fulfillment of the sufficient condition is guaranteed by $\frac{\partial^2 \Pi^e_{i,t+1}}{\partial q^2_{i,t+1}} = 2P'(q_{1,t} + q_{2,t}) - C_i''(q_{i,t+1}) < 0,\ i = 1,2$.

$$JT_{LMA}\left(E^*\right)=\begin{pmatrix} 1-\dfrac{\dfrac{\partial^2\Pi_1\left(q_1^*,q_2^*\right)}{\partial q_1^2}}{2P'(q^*)-C_1''\left(q_1^*\right)} & -\dfrac{\dfrac{\partial^2\Pi_1\left(q_1^*,q_2^*\right)}{\partial q_1\partial q_2}}{2P'(q^*)-C_1''\left(q_1^*\right)} \\ -\dfrac{\dfrac{\partial^2\Pi_2\left(q_1^*,q_2^*\right)}{\partial q_2\partial q_1}}{2P'(q^*)-C_2''\left(q_2^*\right)} & 1-\dfrac{\dfrac{\partial^2\Pi_2\left(q_1^*,q_2^*\right)}{\partial q_2^2}}{2P'(q^*)-C_2''\left(q_2^*\right)} \end{pmatrix}$$

Being the trace and determinant as follows:

$$Tr=Tr\left(JT_{LMA}\left(E^*\right)\right)=2-\left[\dfrac{\dfrac{\partial^2\Pi_1\left(q_1^*,q_2^*\right)}{\partial q_1^2}}{2P'(q^*)-C_1''\left(q_1^*\right)}+\dfrac{\dfrac{\partial^2\Pi_2\left(q_1^*,q_2^*\right)}{\partial q_2^2}}{2P'(q^*)-C_2''\left(q_2^*\right)}\right]$$

$$Det=Det\left(JT_{LMA}\left(E^*\right)\right)=Tr-1+Z_{LMA}$$

with $Z_{LMA}=\dfrac{\dfrac{\partial^2\Pi_1\left(q_1^*,q_2^*\right)}{\partial q_1^2}\dfrac{\partial^2\Pi_2\left(q_1^*,q_2^*\right)}{\partial q_2^2}-\dfrac{\partial^2\Pi_1\left(q_1^*,q_2^*\right)}{\partial q_1\partial q_2}\dfrac{\partial^2\Pi_2\left(q_1^*,q_2^*\right)}{\partial q_2\partial q_1}}{\left[2P'(q^*)-C_1''\left(q_1^*\right)\right]\left[2P'(q^*)-C_2''\left(q_2^*\right)\right]}>0$ if, and only if, (5) is true.

Proposition 3 *In a dynamic duopoly where firms adopt the Local Monopolistic Approximation rule, assuming the condition (5) holds, the convexity of the inverse demand function is a sufficient condition for the local stability of the Cournot-Nash equilibrium.*

Proof. Substituting the above expressions into (8), we can deduce that Schur's conditions are given by:

(i) $1-Tr+Det=Z_{LMA}>0$ if, and only if, condition (5) holds.

(ii) $1+Tr+Det=2Tr+Z_{LMA}$

considering (4), we obtain:

$$Tr=-P''(q^*)\dfrac{2P'(q^*)\,q^*-\left[C_1''\left(q_1^*\right)q_2^*+C_2''\left(q_2^*\right)q_1^*\right]}{\left[2P'(q^*)-C_1''\left(q_1^*\right)\right]\left[2P'(q^*)-C_2''\left(q_2^*\right)\right]}>0\Leftrightarrow P''(q^*)>0$$

Then, the convexity of the function $P(q)$ and the fulfilment of condition (5) assure the second Schur's condition.

(iii) $1-Det=2-(Tr+Z_{LMA})$

considering (4), we obtain:

$$2-(Tr+Z_{LMA})=\frac{\frac{\partial^2\Pi_1^*}{\partial q_1^2}}{2P'(q^*)-C_1''\left(q_1^*\right)}+\frac{\frac{\partial^2\Pi_2^*}{\partial q_2^2}}{2P'(q^*)-C_2''\left(q_2^*\right)}-\frac{\frac{\partial^2\Pi_1^*}{\partial q_1^2}\frac{\partial^2\Pi_2^*}{\partial q_2^2}-\frac{\partial^2\Pi_1^*}{\partial q_1\partial q_2}\frac{\partial^2\Pi_2^*}{\partial q_2\partial q_1}}{\left[2P'(q^*)-C_1''\left(q_1^*\right)\right]\left[2P'(q^*)-C_2''\left(q_2^*\right)\right]}=$$

$$=\frac{\frac{\partial^2\Pi_1^*}{\partial q_1^2}\left[2P'(q^*)-C_2''\left(q_2^*\right)\right]+\frac{\partial^2\Pi_2^*}{\partial q_2^2}\left[2P'(q^*)-C_1''\left(q_1^*\right)\right]-\frac{\partial^2\Pi_1^*}{\partial q_1^2}\frac{\partial^2\Pi_2^*}{\partial q_2^2}+\frac{\partial^2\Pi_1^*}{\partial q_1\partial q_2}\frac{\partial^2\Pi_2^*}{\partial q_2\partial q_1}}{\left[2P'(q^*)-C_1''\left(q_1^*\right)\right]\left[2P'(q^*)-C_2''\left(q_2^*\right)\right]}=$$

$$=\frac{\frac{\partial^2\Pi_1^*}{\partial q_1^2}\left[\frac{\partial^2\Pi_1^*}{\partial q_1^2}-P''(q^*)q_1^*\right]+\frac{\partial^2\Pi_2^*}{\partial q_2^2}\left[\frac{\partial^2\Pi_2^*}{\partial q_2^2}-P''(q^*)q_2^*\right]-\frac{\partial^2\Pi_1^*}{\partial q_1^2}\frac{\partial^2\Pi_2^*}{\partial q_2^2}+\frac{\partial^2\Pi_1^*}{\partial q_1\partial q_2}\frac{\partial^2\Pi_2^*}{\partial q_2\partial q_1}}{\left[2P'(q^*)-C_1''\left(q_1^*\right)\right]\left[2P'(q^*)-C_2''\left(q_2^*\right)\right]}=$$

$$=\frac{\left[\frac{\partial^2\Pi_1^*}{\partial q_1^2}-\frac{\partial^2\Pi_2^*}{\partial q_2^2}\right]^2-P''(q^*)\left[\frac{\partial^2\Pi_2^*}{\partial q_1^2}q_1^*+\frac{\partial^2\Pi_2^*}{\partial q_2^2}q_2^*\right]+\frac{\partial^2\Pi_1^*}{\partial q_1^2}\frac{\partial^2\Pi_2^*}{\partial q_2^2}+\frac{\partial^2\Pi_1^*}{\partial q_1\partial q_2}\frac{\partial^2\Pi_2^*}{\partial q_2\partial q_1}}{\left[2P'(q^*)-C_1''\left(q_1^*\right)\right]\left[2P'(q^*)-C_2''\left(q_2^*\right)\right]}>0 \text{ if } P''(q^*)>0$$

In this case, the convexity of the function $P(q)$ assures the third Schur's condition. □

3.4. *Gradient Rule*

This decisional mechanism is based on marginal profits such that a firm will decide to increase (decrease) its output level, if its marginal profit $\left(\frac{\partial\Pi_i}{\partial q_i}\right)$ is positive (negative) in a certain time period. Formally:

$$q_{i,t+1}-q_{i,t}=\alpha_i(q_{i,t})\frac{\partial\Pi_i\left(q_{1,t},q_{2,t}\right)}{\partial q_i},\ i=1,2$$

The function $\alpha_i(q_{i,t})$ represents firm i's speed of adjustment. A linear relationship, $\alpha_i(q_{i,t})=\alpha_i\, q_{i,t}$, is usually assumed and for the sake of simplicity we will suppose that $\alpha_1=\alpha_2=\alpha>0$. Based on this adjustment mechanism, we obtain the following nonlinear dynamic system:

$$T_G:\begin{cases} q_{1,t+1}=q_{1,t}+\alpha\, q_{1,t}\dfrac{\partial\Pi_1\left(q_{1,t},q_{2,t}\right)}{\partial q_1} \\ q_{2,t+1}=q_{2,t}+\alpha\, q_{2,t}\dfrac{\partial\Pi_2\left(q_{1,t},q_{2,t}\right)}{\partial q_2} \end{cases} \qquad (11)$$

The equilibrium points are defined by the non-negative solutions of the following system[5]:

$$\left.\begin{array}{l} q_1\dfrac{\partial\Pi_1\left(q_1,q_2\right)}{\partial q_1}=0 \\ q_2\dfrac{\partial\Pi_2\left(q_1,q_2\right)}{\partial q_2}=0 \end{array}\right\}$$

[5] The dynamic system (11) has three boundary equilibria: (0, 0), $\left(q_1^m,\ 0\right)$ and $\left(0,q_2^m\right)$ where q_i^m is the monopoly production level. It may easily be proved that these equilibria are unstable.

Thus, the only interior equilibrium point $E^* = \left(q_1^*,\, q_2^*\right)$ is the Cournot-Nash equilibrium. We may formulate the following proposition with regard to the local asymptotic stability of this equilibrium:

Proposition 4 *In a dynamic duopoly where firms follow the gradient rule based on marginal profit, assuming the condition (5) holds, the Cournot-Nash equilibrium is locally asymptotically stable provided that:*

$$\alpha < \alpha_G = -\frac{q_1^*\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2} + q_2^*\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2}}{q_1^* q_2^*\left(\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2}\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2} - \dfrac{\partial^2\Pi_1^*}{\partial q_1\partial q_2}\dfrac{\partial^2\Pi_2^*}{\partial q_2\partial q_1}\right)} - \frac{\sqrt{\left(q_1^*\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2} + q_2^*\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2}\right)^2 - 4q_1^*q_2^*\left(\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2}\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2} - \dfrac{\partial^2\Pi_1^*}{\partial q_1\partial q_2}\dfrac{\partial^2\Pi_2^*}{\partial q_2\partial q_1}\right)}}{q_1^* q_2^*\left(\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2}\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2} - \dfrac{\partial^2\Pi_1^*}{\partial q_1\partial q_2}\dfrac{\partial^2\Pi_2^*}{\partial q_2\partial q_1}\right)} \qquad (12)$$

Proof The Jacobian matrix of T_G evaluated at $E^* = \left(q_1^*,\, q_2^*\right)$ is:

$$JT_G\left(E^*\right) = \begin{pmatrix} 1+\alpha q_1^*\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2} & \alpha q_1^*\dfrac{\partial^2\Pi_1^*}{\partial q_1\partial q_2} \\ \alpha q_2^*\dfrac{\partial^2\Pi_2^*}{\partial q_2\partial q_1} & 1+\alpha q_2^*\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2} \end{pmatrix}$$

where the trace and determinant are:

$$Tr = Tr\left(JT_G\left(E^*\right)\right) = 2 + \alpha\left(q_1^*\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2} + q_2^*\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2}\right)$$
$$Det = Det\left(JT_G\left(E^*\right)\right) = Tr - 1 + Z_G$$

with $Z_G = \alpha^2 q_1^* q_2^*\left(\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2}\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2} - \dfrac{\partial^2\Pi_1^*}{\partial q_1\partial q_2}\dfrac{\partial^2\Pi_2^*}{\partial q_2\partial q_1}\right) > 0$ due to the fulfillment of the condition (5).

The Schur conditions (8) adopt the following expressions:

$$\left.\begin{array}{l} (i)\ 1 - Tr + Det = Z_G > 0 \\ (ii)\ 1 + Tr + Det = 2Tr + Z_G > 0 \\ (iii)\ 1 - Det = 2 - Tr - Z_G > 0 \end{array}\right\}$$

The condition (5) guarantees the first condition.

Substituting and operating on the third condition, we obtain a value α_v such that:

$$2 - Tr - Z_G > 0 \Leftrightarrow -\alpha q_1^* q_2^*\left(\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2}\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2} - \dfrac{\partial^2\Pi_1^*}{\partial q_1\partial q_2}\dfrac{\partial^2\Pi_2^*}{\partial q_2\partial q_1}\right) > q_1^*\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2} + q_2^*\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2} \Leftrightarrow$$

$$\Leftrightarrow \alpha < -\frac{q_1^*\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2} + q_2^*\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2}}{q_1^* q_2^*\left(\dfrac{\partial^2\Pi_1^*}{\partial q_1{}^2}\dfrac{\partial^2\Pi_2^*}{\partial q_2{}^2} - \dfrac{\partial^2\Pi_1^*}{\partial q_1\partial q_2}\dfrac{\partial^2\Pi_2^*}{\partial q_2\partial q_1}\right)} = \alpha_v$$

Thus, assuming (5) is true, if $\alpha < \alpha_v$, the third Schur's condition will be satisfied.

Substituting and operating on the second Schur's condition, we obtain:

$$2Tr + Z_G > 0 \Leftrightarrow 4 + 2\alpha\left(q_1^* \frac{\partial^2 \Pi_1^*}{\partial q_1{}^2} + q_2^* \frac{\partial^2 \Pi_2^*}{\partial q_2{}^2} \right) + \alpha^2 q_1^* q_2^* \left(\frac{\partial^2 \Pi_1^*}{\partial q_1{}^2} \frac{\partial^2 \Pi_2^*}{\partial q_2{}^2} - \frac{\partial^2 \Pi_1^*}{\partial q_1 \partial q_2} \frac{\partial^2 \Pi_2^*}{\partial q_2 \partial q_1} \right) > 0$$

According to the condition (5) and the assumptions of the model, the parabola $y(\alpha) = q_1^* q_2^* \left(\frac{\partial^2 \Pi_1^*}{\partial q_1{}^2} \frac{\partial^2 \Pi_2^*}{\partial q_2{}^2} - \frac{\partial^2 \Pi_1^*}{\partial q_1 \partial q_2} \frac{\partial^2 \Pi_2^*}{\partial q_2 \partial q_1} \right) \alpha^2 + 2\left(q_1^* \frac{\partial^2 \Pi_1^*}{\partial q_1{}^2} + q_2^* \frac{\partial^2 \Pi_2^*}{\partial q_2{}^2} \right) \alpha + 4,$ is U-shaped and its vertex is $\left(\alpha_v, y\left(\alpha_v\right)\right)$ with $y\left(\alpha_v\right) < 0$. Thus, the parabola cuts the abscissa axis at two points $\left(\alpha_1^*, 0\right)$ and $\left(\alpha_2^*, 0\right)$ being:

$$\alpha_1^* = \alpha_v - \frac{\sqrt{\left(q_1^* \frac{\partial^2 \Pi_1^*}{\partial q_1{}^2} + q_2^* \frac{\partial^2 \Pi_2^*}{\partial q_2{}^2} \right)^2 - 4 q_1^* q_2^* \left(\frac{\partial^2 \Pi_1^*}{\partial q_1{}^2} \frac{\partial^2 \Pi_2^*}{\partial q_2{}^2} - \frac{\partial^2 \Pi_1^*}{\partial q_1 \partial q_2} \frac{\partial^2 \Pi_2^*}{\partial q_2 \partial q_1} \right)}}{q_1^* q_2^* \left(\frac{\partial^2 \Pi_1^*}{\partial q_1{}^2} \frac{\partial^2 \Pi_2^*}{\partial q_2{}^2} - \frac{\partial^2 \Pi_1^*}{\partial q_1 \partial q_2} \frac{\partial^2 \Pi_2^*}{\partial q_2 \partial q_1} \right)}$$

$$\alpha_2^* = \alpha_v + \frac{\sqrt{\left(q_1^* \frac{\partial^2 \Pi_1^*}{\partial q_1{}^2} + q_2^* \frac{\partial^2 \Pi_2^*}{\partial q_2{}^2} \right)^2 - 4 q_1^* q_2^* \left(\frac{\partial^2 \Pi_1^*}{\partial q_1{}^2} \frac{\partial^2 \Pi_2^*}{\partial q_2{}^2} - \frac{\partial^2 \Pi_1^*}{\partial q_1 \partial q_2} \frac{\partial^2 \Pi_2^*}{\partial q_2 \partial q_1} \right)}}{q_1^* q_2^* \left(\frac{\partial^2 \Pi_1^*}{\partial q_1{}^2} \frac{\partial^2 \Pi_2^*}{\partial q_2{}^2} - \frac{\partial^2 \Pi_1^*}{\partial q_1 \partial q_2} \frac{\partial^2 \Pi_2^*}{\partial q_2 \partial q_1} \right)}$$

It is verified that $0 < \alpha_1^* < \alpha_v < \alpha_2^*$, and:

$$2Tr + Z > 0 \Leftrightarrow \begin{cases} \alpha < \alpha_1^* \\ \alpha > \alpha_2^* \end{cases}$$

Hence, assuming (5) is true, the three Schur's conditions will be satisfied if $\alpha < \alpha_1^*$. Given $\alpha_G = \alpha_1^*$, the proposition is proved. □

We may note it is necessary to impose a threshold for the adjustment speed parameter when firms follow the gradient rule in order to guarantee the local stability of the Nash equilibrium. Above this threshold, the Nash equilibrium loses stability through a flip bifurcation and complex dynamic behaviors appear (see among others Andaluz and Jarne 2016; Askar 2020). It is important to note that the definition of this threshold requires the fulfilment of the condition (5).

This threshold α_G could be interpreted as preventing firms from overreacting to market signals. The expression of the threshold will depend on the derivatives of the profit functions, which are determined by the demand and cost structure.

4. Conclusions

One of the most questioned aspects of Cournot's model has been the adjustment process followed by the quantities until they converge to equilibrium.

In recent years, numerous contributions have emerged whose objective is the dynamic analysis of the resulting equilibria in oligopolistic markets where firms have bounded rationality in a context of discrete time.

Generally, the existing works in this line has conventionally considered particular specifications of the elements defining the structure of the market, such as demand and costs. This paper has sought to generalize these findings by analyzing the stability conditions of the Cournot-Nash equilibrium in a duopoly with homogeneous product, under different expectations schemes and assuming a general form for the demand and cost functions.

Based on the analysis carried out, we were able to deduce that the condition that ensures the stability of the Nash equilibrium derived from the tâtonnement process in a static model, is a determining factor in a dynamic context, being this condition less decisive the more bounded the rationality of the firms.

On the one hand, it is a necessary and sufficient condition to guarantee the local dynamic stability of the Cournot-Nash equilibrium, provided that players make their decisions according to a scheme of naïve or adaptive expectations.

When firms have limited knowledge of the environment, they make their decisions according to expectation schemes that imply a lower degree of rationality, such as the Local Monopolistic Approximation and the gradient rule based on marginal profit.

In the first case, assuming the fulfillment of the condition that ensures the stability of the Nash equilibrium in a static model, the convexity of the inverse demand function arises as a sufficient condition to ensure the local stability of the equilibrium. On the other hand, when both firms follow the gradient rule based on the marginal profits, the condition of stability derived in a static context is necessary to define a critical value for the firms' speed of adjustment, which determines the local stability of the Nash equilibrium. Above this threshold, the Nash equilibrium becomes unstable through a cascade of flip bifurcations and complex dynamics may appear.

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